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Efficient Computation of Most Probable States for Communication Networks with Multimode Components

CHE-LIANG YANG AND PETER KUBAT

Abstract—**A general multimode model for jointly assessing the performance and reliability (performability) of a communication network is considered. The performability is computed by averaging the performance over the most probable network states. An efficient algorithm, superior to all previous algorithms in both speed and flexibility, is proposed to enumerate the most probable network states. The algorithm keeps generating the network states one by one, in order of decreasing probability, until some stopping rule (if specified) is met.**

I. Introduction

In recent years, efforts [1]–[3] have been devoted to jointly assessing the performance and reliability of a communication/computer network. In this approach, all the possible states of the network are enumerated, steady-state probability and network performance calculated for each state, and the overall performance (performability) calculated as the average network performance in the steady state. Due to the large (exponential) size of the state space, in practice, such assessment is limited to very small networks.

To avoid the pitfall of the exponential size of the state space, Li and Silvester [4] proposed an approximation—to calculate the global performance over the most probable states of the network only. Since the network operates in these states most of the time, this suggests a good approximation of the network performance. In networks where component reliability is high, only a small number of states dominate the state space in terms of probability, and thus the calculation of the global performance is fast. The efficiency of the algorithm for generating most probable states [4] was improved by Lam and Li [6].

In a recent article, Chiou and Li [5] generalized the approach and introduced an algorithm for approximately calculating the reliability of communication networks with multimode components. Here the performance degradation of individual components is characterized by different modes. Chiou and Li's algorithm, named ORDER-M, enumerates m network states s_k, $k = 1, 2, \cdots, m$, in order of decreasing probability and then averages the network performance over these m most probable states. For the network with n components and N modes for each component, the complexity of their algorithm is $O(n^2Nm + nNm \log m)$. The algorithm requires the selected value of m to be predetermined.

In order to achieve a prescribed so-called $(1 - \epsilon)$ coverage of the entire sample space Ω, Chiou and Li would have to first guess the value of m, then run the algorithm to generate m most probable states, and finally check to see if m will satisfy the coverage requirement that $\Sigma_{k=1}^{m} P(s_k) \geq 1 - \epsilon$. If the requirement is not satisfied, the larger value of m is selected, and the algorithm will be again executed from the start and so on.

In this paper, we present a new efficient algorithm that can enumerate the most probable states s_k of the network with multimode components, in order of decreasing probability. The algorithm keeps generating states, one by one, in every $O(nN)$ operations, until any suitable stopping criterion (if specified) is met. To identify Ω_ϵ, the minimal set of most probable states that satisfies $(1 - \epsilon)$ coverage requirement, we can run the algorithm until the probability sum of the states generated so far is no less than $(1 - \epsilon)$. It is shown that the complexity is $O(n \cdot N \cdot |\Omega_\epsilon|)$. Our algorithm does not need to guess the size of Ω_ϵ in advance, and it is at least n times faster than the one developed by Chiou and Li [5].

II. The Model—Assumptions and Notation

Consider a communication network which is composed of n components (or, units) numbered by $1, 2, \cdots, n$. For each $i = 1, 2, \cdots, n$,

1) unit i operates in one of $N_i \geq 1$ modes, called $M_{i,1}, M_{i,2}, \cdots, M_{i,N_i}$;

2) unit i operates in mode $M_{i,j}$ with probability $p_{i,j} > 0$, $\Sigma_{j=1}^{N_i} p_{i,j} = 1$, independent of other units.

Without loss of generality, we will assume that for each $i = 1, 2, \cdots, n$,

$$p_{i,1} \geq p_{i,j}, \quad j = 2, \cdots, N_i. \tag{1}$$

Let the n vector $x = (x_1, x_2, \cdots, x_n)$ represent a possible state of the network units such that $x_i = j$, $1 \leq j \leq N_i$ if unit i is in mode $M_{i,j}$. Clearly, there are $\Pi_{i=1}^{n} N_i$ possible states for the network. Let $P(x)$ denote the probability that the network is in state x. Thus, $P(x) = \Pi_{i=1}^{n} p_{i,x_i}$. Define the minimal $(1 - \epsilon)$ covered space Ω_ϵ as

$$\Omega_\epsilon = \left\{ s_k \,|\, k = 1, 2, \cdots, m, \sum_{k=1}^{m-1} P(s_k) < 1 - \epsilon \leq \sum_{k=1}^{m} P(s_k) \right\}$$

where the network states s_k, $k = 1, 2, \cdots, \Pi_{i=1}^{n} N_i$ are assumed to be indexed so that $P(s_1) \geq P(s_2) \geq P(s_3) \geq \cdots$.

To describe the algorithm which finds s_i one by one, in order of decreasing probability, we need the following definitions (see [7] for details).

A (*directed*) *tree* is a directed graph such that there exists exactly one vertex, called the *root*, which no edges enter and there is a unique directed path from the root to every vertex.

Let $G = (V, E)$ be a tree where V and E represent the vertex set and edge set, respectively. If the directed edge (u, v) is in E, then u is called the *father* of v, and v is a *son* of u. If there is a directed path from u to v, then u is an *ancestor* of v, and v is a *descendant* of u. A vertex with no descendants is called a *leaf*; otherwise, this vertex is called an *internal vertex*. A vertex u and all its descendants, denoted by T_u, is called a *subtree* of G. The *level* of a vertex u is the length (number of edges) of the path from the root to this vertex.

Paper approved by the Editor for Computer Communications Theory of the IEEE Communications Society. Manuscript received February 15, 1987; revised February 15, 1988 and November 2, 1988. This paper was presented at the Twenty-First Annual Conference on Information Sciences and Systems, The Johns Hopkins University, Baltimore, MD, March 25–27, 1987.

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IEEE Log Number 8926978.

0090-6778/89/0500-0535$01.00 © 1989 IEEE

Clearly, the root of G is the unique vertex that is at level 0. The *height* of a tree is the length of the longest path from the root to a leaf.

An *ordered tree* is a tree in which the sons of each vertex are ordered. When drawing an ordered tree, we assume that the sons of each vertex are ordered from left to right. T_v is called the ith subtree of u if v is the ith son of u.

III. An Algorithm for Identifying Ω_ϵ

First, we transform the problem of identifying Ω_ϵ to a tree search problem. Given N_i and $p_{i,j}$ for $i = 1, \cdots, n$ and $j = 1, \cdots, N_i$, define an ordered tree G with height n so that each vertex at level l, $0 \le l \le n-1$, has exactly N_{l+1} sons and each vertex at level n is a leaf. Clearly, this tree has $\Pi_{i=1}^{n} N_i$ leaves. Define the *address* of a vertex u at level l, $1 \le l \le n$, to be an l vector $(y_1, y_2, \cdots, y_l)$ where $y_i = j$ if for the unique path, say $a_0(root), a_1, \cdots, a_{l-1}, a_l(u)$, from the root to u, a_i is the jth son of a_{i-1}. The *weight* of vertex u, denoted as $\omega(u)$, is defined to be $\Pi_{i=1}^{l} p_{i,y_i}$. Note that there is one-to-one correspondence between the address of a leaf in G and a state in Ω. Thus, the problem of finding Ω_ϵ is equivalent to identifying the addresses of the heaviest leaves in G such that their weight sum is greater than or equal to $1 - \epsilon$.

Example 1: Consider a 3-unit system whose table of N_i and $p_{i,j}$ is shown in Table I. The corresponding ordered tree G is depicted in Fig. 1 where associated with each leaf is the address and weight. Observe that there are 12 leaves in G and each leaf corresponds to exactly one state of the system. □

To solve this problem, our basic strategy is quite simple—first identify the address of the heaviest leaf in G, then identify that of the second heaviest leaf, and so on until the weight sum of all leaves identified so far is greater than or equal to $1 - \epsilon$. In other words, the address of the mth heaviest leaf of G will be identified at the mth iteration of the algorithm. Note that since G is fully defined by the table of N_i and $p_{i,j}$, we can conceive the structure of the whole G without having to create real tree structures to represent it.

Lemma 1: For a vertex u of G at level l, the heaviest leaf in T_u has weight

$$\omega(u) \cdot \prod_{i=l+1}^{n} p_{i,1}.$$

Proof: This follows directly from (1) and the definition of G. □

The basic operation of the algorithm is described as follows by induction. Suppose that we are at the beginning of the mth iteration of the algorithm and that we have built a subgraph G' of G with the vertex set corresponding to those leaves identified (in the first $m-1$ iterations) and their ancestors. Furthermore, suppose that each internal vertex u (at level l) of G' is associated with an array $W_u[1], \cdots, W_u[N_{l+1}]$ such that $W_u[i]$ indicates the weight of the heaviest remaining (that is, not yet identified) leaf in the ith subtree of u in G. Let $W_u[i] = 0$ if all the leaves in the ith subtree of u in G have been identified. With this, we can now identify the heaviest remaining leaf (in this case, the mth heaviest leaf) of G by proceeding from the root of G' and determining to visit the kth son of the currently being visited vertex u at level $l < n$ if $W_u[k] = \max(W_u[1], \cdots, W_u[N_{l+1}])$. If the kth son of u has not yet been created in G', then before visiting it, we have to first create a vertex u' as the kth son of u, establish a link from u to u', and associate u' with a newly created array $(W_{u'}[1], \cdots, W_{u'}[N_{l+2}])$ where at this point, we can simply let $W_{u'}[i] = \omega(u') \cdot p_{l+1,i} \cdot \Pi_{j=l+2}^{n} p_{j,1}$ because of Lemma 1 and the fact that no leaf in $T_{u'}$ of G has been identified. When finally a leaf (that is, a vertex at level n), say v, is reached, we can be sure that v corresponds to the heaviest remaining leaf of G. Suppose that $a_0(root), a_1, \cdots, a_{n-1}, a_n(v)$ is the unique path from the root to v and that $(y_1, y_2, \cdots, y_n)$ is the address of v. (The address of v can be determined as the algorithm traverses from the root to v). Note that after the leaf v has been identified, it is not necessary to update $W_u[1], W_u[2], \cdots$ for every internal vertex u in G' because only $W_{a_i}[y_{i+1}]$, $i = 0, \cdots, n-1$, can be affected. (To see this, observe that the content of each $W[i]$ of a vertex should remain unchanged if v is not in the ith subtree of this vertex.) We can easily update $W_{a_i}[y_{i+1}]$, $i = 0, \cdots, n-1$ by setting $W_{a_{n-1}}[y_n] = 0$ and $W_{a_i}[y_{i+1}] = \max(W_{a_{i+1}}[1], \cdots, W_{a_{i+1}}[N_{i+2}])$, $i = n-2, \cdots, 0$, when backtracking, vertex by vertex, from v to the root of G'. As soon as the backtracking is completed, the algorithm is ready for the $(m+1)$th iteration because the induction hypothesis of G' for the next iteration is satisfied now.

At the beginning of the first iteration where no leaf of G has yet been identified and, therefore, G' is empty, we can first create a vertex as the root of G', associate it with a new array $(W_{root}[1], \cdots, W_{root}[N_1])$ such that $W_{root}[i] = p_{1,i} \cdot \Pi_{j=2}^{n} p_{j,1}$, and then follow the general instructions as described above.

Example 2: Consider again the 3-unit system in Example 1 where the corresponding ordered tree G was depicted in Fig. 1. To identify $\Omega_{\epsilon=0.35}$, the algorithm grows a subgraph G' of G step by step. (Note that the algorithm does not create G. We show it here only for illustration.) In Figs. 2(a)–3(a), we show, respectively, the G' at iterations 1–2 immediately before backtracking. We also show in Figs. 2(b)–3(b), respectively, the G' at iterations 1–2 after backtracking. The unique path traversed from the root to a leaf at each iteration is highlighted by double-circled nodes. Only 4 iterations are required in this case and the algorithm generates (in order) outputs $\Omega_{\epsilon=0.35} = \{(1, 1, 1)(1, 3, 1)(2, 1, 1)(1, 2, 1)\}$. □

A complete description of the algorithm is given next. For a currently being visited vertex u of G' at level l with weight ω and address $(y_1, y_2, \cdots, y_l)$ where it is known that T_u in G has a leaf (referred to as v when found) which is the heaviest among all the remaining leaves of G, we now show a *recursive* procedure SEARCH(u, l, ω, NEW). This procedure can grow G' to build the unique path, say $a_l(u), a_{l+1}, \cdots, a_n(v)$, from u to v, identify the address $(y_1, y_2, \cdots, y_n)$ of v, and then update $W_{a_i}[y_{i+1}]$, $i = n-1, \cdots, l$. It finally returns a value NEW which is the weight of the heaviest remaining leaf

TABLE I
THE DATA FOR THE SYSTEM IN EXAMPLE 1

unit i	N_i	$p_{i,1}$	$p_{i,2}$	$p_{i,3}$
1	2	0.7	0.3	
2	3	0.5	0.2	0.3
3	2	0.8	0.2	

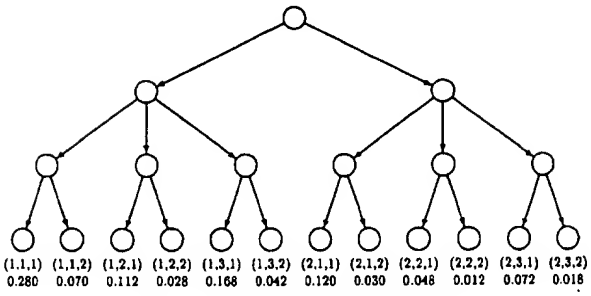

Fig. 1. The ordered tree G for Example 1.

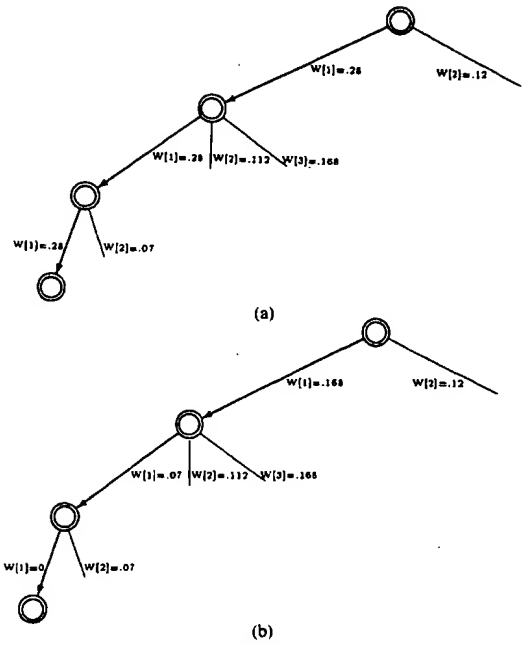

Fig. 2. (a) G' at iteration 1 after backtracking. (b) G' at iteration 1 after backtracking.

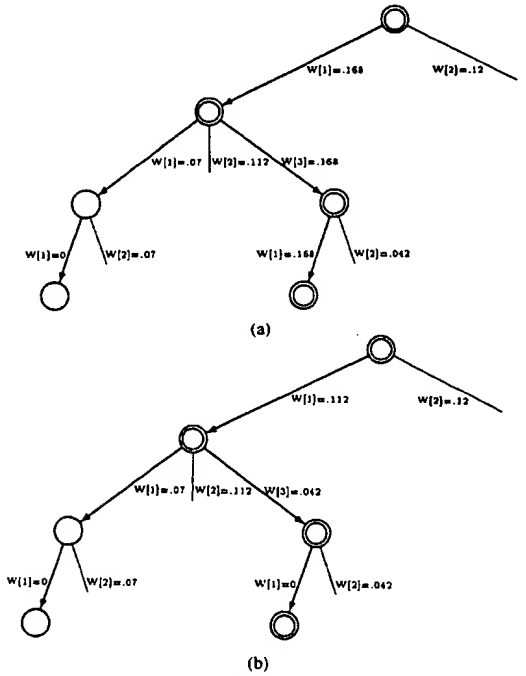

Fig. 3. (a) G' at iteration 2 after backtracking. (b) G' at iteration 2 after backtracking.

among T_u of G after v has been found. There are two possible cases.

Case I: $l = n$ (i.e., u is a leaf), The procedure SEARCH(u, l, ω, NEW) will simply print the contents of $(y_1, y_2, \cdots, y_n)$ and the return with NEW = 0.

Case II: $l < n$ (i.e., u is an internal vertex). If u is visited for the first time, procedure SEARCH(u, l, ω, NEW) will first create and initialize $W_u[i] = \omega \cdot p_{l+1,i} \cdot \Pi_{j=l+2}^{n} p_{j,1}$ for each $i = 1, \cdots, N_{l+1}$. Choose k such that $W_u[k] = \max(W_u[1], \cdots, W_u[N_{l+1}])$. If the kth son of u has not yet been created in G', create vertex u' as the kth son of u, and establish a link from u to u'. The procedure then proceeds to visit the kth son of u, let $y_{l+1} = k$, recursively call SEARCH(u', $l + 1$, $\omega \cdot p_{l+1,k}$, NEW') to build the path $a_{l+1}(u')$, a_{l+2}, $\cdots$, $a_n(v)$ from u' to v, identify the address $(y_1, y_2, \cdots, y_n)$ of v, update $W_{a_i}[y_{i+1}]$, $i = n - 1, \cdots, l + 1$, and finally return a value NEW' which indicates the weight of the heaviest remaining leaf among the kth subtree of u in G after v has been identified. As soon as SEARCH(u', $l + 1$, $\omega \cdot p_{l+1,k}$, NEW') is returned, the procedure will update $W_u[k]$ with NEW' and return with NEW = max ($W_u[1]$, $\cdots$, $W_u[N_{l+1}]$).

With the procedure SEARCH(u, l, ω, NEW), the main routine for printing Ω_ϵ only needs to create a vertex as the root, then repreatedly call SEARCH(*root*, 0, 1, NEW) until the weight sum of all the leaves identified so far becomes greater than or equal to $1 - \epsilon$. A formal description of the procedure SEARCH and main routing is as follows:

Procedure SEARCH(u, l, ω, NEW)
Step 1) **if** $l = n$ **then**
begin
print $(y_1, y_2, \cdots, y_n)$;
SUM $\leftarrow$ SUM + ω;
NEW $\leftarrow$ 0;
return
end;
Step 2) **if** u has not been visited before **then**
create and let $W_u[i] = \omega \cdot p_{l+1,i} \cdot P_1[l+1]$ for $i = 1, \cdots, N_{l+1}$;
Step 3) choose k such that $W_u[k] = \max(W_u[1], \cdots, W_u[N_{l+1}])$;
Step 4) **if** the kth son of u has not yet been created **then**
begin
create a vertex u' as the kth son of u;
establish a link from u to u'
end;
Step 5) $y_{l+1} \leftarrow k$;
visit u' (that is, the kth son of u) via the link established before;
call SEARCH(u', $l+1$, $\omega \cdot p_{l+1,k}$, NEW');
Step 6) $W_u[k] \leftarrow$ NEW';
NEW $\leftarrow$ max ($W_u[1]$, $\cdots$, $W_u[N_{l+1}]$);
Step 7) **return**.

Main Routine
Input: ϵ and $p_{i,j}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, N_i$;
Output: Ω_ϵ;
Step 0) create an n vector $(y_1, y_2, \cdots, y_n)$;
SUM $\leftarrow$ 0;
$P_1[n] \leftarrow 1$;
for $l = n - 1$ **to** 0 **do** $P_1[l] \leftarrow p_{l+1,1} \cdot P_1[l + 1]$;
create a vertex root;
Step 1) **repeat**
visit root;
call SEARCH(*root*, 0, 1, NEW)
until SUM $\geq 1 - \epsilon$;
Step 2) stop.

Note that the value of variable SUM in the algorithm corresponds to the accumulated probability of the most probable states enumerated so far. Observe that the complexity is dominated by *Step 1)* in the main routine. At each iteration of *Step 1)*, $n + 1$ vertices of G' at different levels (corresponding to the vertices on the unique path from the root to the heaviest remaining leaf in G) are visited and/or created where the algorithm spends $O(N_{l+1})$ operations on the vertex at level $l < n$ and spends $O(n)$ operations on the vertex at

level n for printing an n-vector. Since *Step 1)* has $|\Omega_\epsilon|$ iterations, the time complexity of the algorithm is

$$O\left(|\Omega_\epsilon|n+|\Omega_\epsilon| \cdot \sum_{i=1}^{n} N_i\right)$$

$$= O\left(|\Omega_\epsilon| \cdot \sum_{i=1}^{n} N_i\right) \le O(|\Omega_\epsilon| nN)$$

where $N = \max(N_1, N_2, \cdots, N_n)$.

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Effect of Ordering the Codebook on the Efficiency of the Partial Distance Search Algorithm for Vector Quantization

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Abstract—**Recently, Bei and Gray have used a partial distance search algorithm which reduces the computational complexity of the minimum distortion encoding for vector quantization. In the present correspondence, the effect of ordering the codevectors on the computational complexity of the algorithm is studied. It is shown that the computational complexity of this algorithm can be reduced further by ordering the codevectors according to the sizes of their corresponding clusters.**

I. Introduction

Vector quantization has become recently very popular in a number of areas such as speech coding, image coding and speech recognition [1], [2]. However, the utilization of vector quantization is severely limited by the computational complexity of the encoding process. The minimum distortion encoding of a test vector, using the conventional full-search algorithm for a codebook of size N ($=2^{KR}$, K being vector dimension and R the bit rate) requires N vector-distance computations. For the squared-error distance measure, the number of multiplications required per sample for the encoding is N, i.e., the size of the codebook, which increases exponentially with dimension K and bit rate R. The encoding process, therefore, becomes prohibitively expensive for large-size codebooks. In order to reduce the computational complexity of the minimum distortion search in a vector quantization encoding process, the partial distance search method has been recently proposed [3], [4]. In this method, a codevector can be rejected on the basis of partial distance, i.e., without completing the total distance computation.

In the present paper, our aim is to study the effect of ordering the codevectors on the savings offered by the partial distance search. To maximize this savings, we propose an ordering of codevectors according to the sizes of their corresponding clusters. In addition, we note that the codebooks obtained at the end of the training process using the clustering algorithms such as the Linde–Buzo–Gray algorithm [5] have arbitrary orderings and are never guaranteed to be arranged in the favorable order. We demonstrate this and show that an explicit ordering of the codevectors as proposed here is advantageous for deriving more computational savings from the partial distance method.

II. Algorithm

Let $C = \{c_i, i = 1, \cdots, N\}$ be a codebook of size N where $c_i = (c_{ij}, j = 1, \cdots, K)$ is a K dimensional vector. For a given test vector $X = (x_j, j = 1, \cdots, K)$, it is required to find the minimum distortion codevector $q(X)$ from the codebook C under the squared-error distortion measure defined as

$$d(X, c_i) = \sum_{j=1}^{K} (x_j - c_{ij})^2.$$

The basic structure of the partial distance search algorithm [4] is as follows:

$d_{\min} = \infty$ (a very large number).

Loop A: For $i = 1, \cdots, N$
  $d = 0$
Loop B: For $j = 1, \cdots, K$
    $d = d + (x_j - c_{ij})^2$
    if $d > d_{\min} \rightarrow$ ***next i*** (exit condition)
  next j
  $d_{\min} = d$
  min $= i$
next i.

It can be observed that the partial-distance search algorithm gains computation saving over the full search algorithm because of the provision for a premature exit from Loop B on satisfying the condition $d > d_{\min}$ (called the exit condition henceforth) before the completion of the distance computation $d(X, c_i)$.

For a given test vector X, the maximum saving is obtained when the exit condition is satisfied as early as possible inside Loop B for each c_i. This in turn is facilitated by the condition that the minimum possible value of $d_{\min}$ for a given X is obtained earliest in Loop A. Since $d_{\min}$ has the minimum possible value for a given X only after the actual nearest codevector of X has been checked, it is possible to obtain the maximum saving in computation by having the first codevector as the nearest codevector of X, i.e., $c_1 = q(X)$. This is obviously not practical since prior knowledge of $q(X)$ obviates the need for search. However, a probabilistic solution can be given to improve the computation saving.

Paper approved by the Editor for Quantization, Speech/Image Coding of the IEEE Communications Society. Manuscript received April 30, 1987; revised December 18, 1987.

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IEEE Log Number 8926995.

Singapore ICCS/ISITA '92

ASYMPTOTIC PERFORMANCE FOR ORTHOGONAL SIGNALLING ON FADING, NON-GAUSSIAN CHANNELS

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INTRODUCTION

The problem of detecting one out of M, possibly fading, signals immersed in additive noise can be posed as an $M-$ary hypothesis testing problem: the received signal, after baseband demodulation, is forwarded to the decision circuits, which are eventually to decide which one, between the M alternative hypotheses, is actually in force. In other words, if the $i-$th alphabet symbol is transmitted, namely under hypothesis H_i, then the baseband equivalent of the received signal is

$$\tilde{r}(t) = A\exp(j\theta)\tilde{p}_i(t) + \tilde{n}(t) \qquad i = 1, \ldots, M \qquad (1)$$

wherein $\tilde{p}_i(\cdot)$ is the complex envelope of the $i-$th waveform, $A\exp(j\theta)$ is a complex gain accounting for the possible channel fading, and $\tilde{n}(t)$ is the complex envelope of the noise, which is typically modelled as a complex, possibly correlated stochastic process. For equally likely, equi-energy signals, the detection scheme (see figure 1) consists of a bank of M filters, each matched to one of the admissible waveforms, followed by an envelope detector: the maximum of these envelopes is henceforth used for making the final decision [1].

The rationale for adopting such a scheme is two-fold: on one hand, in fact, it turns out to achieve minimum error probability - uniformly with respect to the probability density function (pdf) of A - for the case that the noise can be modelled as a white, complex Gaussian process and the signal phase θ is uniformly distributed in $(-\pi, \pi)$. On the other, for orthogonal signal set with non-fluctuating amplitude, this detector is also asymptotically optimum for increasingly high signal set size M, in the sense that arbitrarily reliable communication can be achieved, provided the signal-to-noise per bit exceeds a threshold level, often referred to as Shannon limit [1]. Unfortunately, in many situations of practical interest the Gaussian assumption for the noise is no longer valid. Relevant deviations from the Gaussian law, especially in the high-amplitude tail of the measured noise distribution, are observed both for man-made noise [2,3]

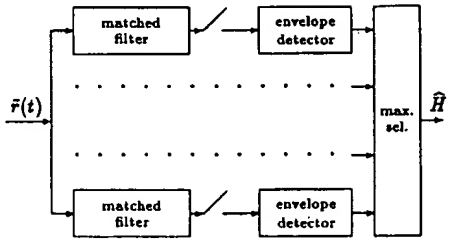

Figure 1: Scheme of the envelope detector

and for atmospheric noise [3,4,5]: in other words, such a disturbance turns out to be *impulsive*, in the sense that noise spikes occur with higher probability than predicted by the Gaussian pdf. The classical envelope detector is no longer optimum in this case in either sense specified above: on one hand, in fact, it is not ensured to achieve minimum error probability for finite M, on the other its performance may not attain the Shannon limit for $M \to \infty$. As to the former point, recent studies [6,7,8] demonstrated that, if the impulsive noise is modelled as a compound-Gaussian process, then the classical envelope detector is still optimum, for any set of equally likely, equi-energy signals and for any amplitude fluctuation law: in other words, such a detector is *canonical* with respect to both the noise marginal distribution - provided it belongs to the class of compound-Gaussian pdf's - and the pdf of A.

This paper is aimed at investigating the latter point, namely at studying the impact of the noise distribution on the limit performance: in fact, even though the situation $M \to \infty$ is a merely ideal one, since it assumes communication with no finite-bandwidth constraint, it is of relevant theoretical interest to assess whether increased signal space dimensionality - and, hence, larger transmission bandwidth - can be still traded at will for enhanced communication reliability.

More precisely, in next section, after shortly reviewing

the noise model, we summarise some relevant results about detection in compound-Gaussian noise; in the last section we outline analytical procedures for evaluating the error probability for $M-$ary orthogonal signalling; in particular, we give a general relationship, holding true for any noise pdf as well as signal amplitude fluctuation law, showing that even in non-fading channels noise spikyness dramatically affects the limit performance.

DETECTION IN IMPULSIVE NOISE

The optimum - in the usual sense of achieving minimum error probability - decision rule for discriminating between M equally likely hypotheses is well known to be the Maximum Likelihood (ML) rule; therefore, the design of the optimum detector requires the specification of the received signal under the M alternative hypotheses - or, equivalently, of the noise process - in order to calculate the relevant likelihood functionals [1]. Consequently, the need arises for a statistical model of the noise baseband equivalent $\tilde{n}(t)$: generally speaking, such a model should, on one hand, be mathematically tractable, on the other, it should be physically grounded, namely it should represent a faithful mathematical description of the physical phenomena that give rise to the noise.

A class of processes that fulfill both requirements is that of the exogenous processes, wherein the noise is regarded as the product of two mutually independent components, namely

$$\tilde{n}(t) = s(t)\tilde{g}(t) \tag{2}$$

The process $\tilde{g}(t)$ is a zero-mean, complex, Gaussian, possibly correlated process, whilst $s(t)$ is a real, non-negative process, whose average decorrelation time is heretoafter assumed to be much longer than the decorrelation time of $\tilde{g}(t)$. This model was first introduced and validated by Hall [4] with reference to atmospheric noise and subsequently widely referred to as a suitable one for many sources of impulsive noise [3,5]. Notice also that if the coherence time of $s(t)$ is much longer than the signalling interval, then the noise process (2), as observed at the receiver, degenerates into a spherically symmetric process, namely into the product of the Gaussian process $\tilde{g}(t)$ times a *random constant* s, i.e.:

$$\tilde{n}(t) = s\tilde{g}(t) \tag{3}$$

This approximation results in considerable mathematical simplifications, since the process (3) can be shown to be amenable to a complete statistical characterisation based solely upon knowledge of its marginal pdf and covariance function [6].

Unfortunately, not all the marginal pdf's are compatible with the representations (2) and (3); in fact, notice that, by the total probability law, the common marginal pdf of the noise quadrature components, $f_n(x)$ say, should identically satisfy the relationship

$$f_n(x) = \int_0^\infty \frac{1}{\sqrt{2\pi s^2\sigma^2}} \exp\left(-\frac{x^2}{2s^2\sigma^2}\right) f(s)\, ds \tag{4}$$

wherein σ^2 denotes the power of $\tilde{g}(t)$ as well as of the overall noise process: in the above equation, indeed, we assumed, without loss of generality, that s possesses unit mean square value. Therefore, in order that the process $\tilde{n}(t)$ be admissible as a compound-Gaussian one, it is necessary and sufficient that a pdf $f(s)$ exist satisfying the integral equation (4).

However, all the most credited models for the noise marginal pdf can be shown to be admissible as compound Gaussian processes; among them we cite [6]

- the Generalised Gaussian, namely

$$f_n(x) = \frac{\nu}{2a\Gamma(1/\nu)} \exp\left[-\left(\frac{|x|}{a}\right)^\nu\right], \tag{5}$$

  where $a > 0$ is related to the variance σ^2 as $a^2 = \sigma^2 \frac{\Gamma(1/\nu)}{\Gamma(3/\nu)}$ and $\nu > 0$ is a shape parameter. The admissibility range of the shape parameter of this pdf has been shown to be $0 < \nu \leq 2$.

- the Generalised Laplace, namely

$$f_n(x) = \frac{a}{\sqrt{\pi}\Gamma(\nu)} \left(\frac{a|x|}{2}\right)^{\nu-1/2} K_{\nu-1/2}(a|x|), \tag{6}$$

  where $\nu > 0$ is again a shape parameter, whilst $a > 0$ is related to the variance σ^2 as $a^2 = \frac{\sigma^2}{2\nu}$. Unlike the previous pdf, the Generalised Laplace is admissible in the whole range of its shape parameter.

- the Generalised Cauchy, namely

$$f_n(x) = \frac{a^{2\nu}\Gamma(\nu+1/2)}{\sqrt{\pi}\Gamma(\nu)} (a^2+x^2)^{-\nu-1/2}, \tag{7}$$

  where $a > 0$ is still a scale parameter, while $\nu > 0$ is a shape parameter: such a distribution is admissible for any ν.

- the Middleton Class-A, namely

$$f_n(x) = \sum_i \frac{\epsilon_i}{\sqrt{2\pi s_i^2\sigma^2}} \exp\left(-\frac{x^2}{2s_i^2\sigma^2}\right) \tag{8}$$

  wherein s_i are discrete values that s can take on with probabilities ϵ_i. In Middleton's Class-A model these parameters assume the form

$$\epsilon_i = e^{-\nu}\frac{\nu^i}{i!} \qquad s_i^2 = \frac{i/\nu+\lambda}{1+\lambda}; \quad i = 0, 1, \ldots \tag{9}$$

  where ν is a positive shape parameter - also referred to as *impulsive index* - and λ is the ratio of the power of the Gaussian component of the noise to that of the impulsive (Poisson) component.

Based on the model described above, we can proceed to the solution of the $M-$ary detection problem. By applying the usual procedures of the statistical detection theory, it can be shown that, insofar as the noise is white, the test admits the dot products

$$\left| \int \tilde{r}(t)\tilde{p}_i^*(t)\, dt \right| \qquad i = 1, \ldots, M$$

as sufficient statistics [7,8], which leads to the optimum test

$$\text{if } \left| \int \tilde{r}(t)\tilde{p}_i^*(t)\, dt \right| > \left| \int \tilde{r}(t)\tilde{p}_k^*(t)\, dt \right| \quad \forall k \neq i \qquad (10)$$

then choose H_i

namely to the conventional detector of figure 1. We stress here that the detector (10) is *canonically* optimum irrespective of the pdf $f(s)$ - and hence, of the noise marginal pdf - in the class of spherically symmetric impulsive noise as well as of the fading law $f(A)$ and the transmitted equi-energy-signals constellation: this is a straightforward, although not trivial, consequence of the noise model we adopted and, especially, of the assumption that the process $s(t)$ is approximately constant over the signalling interval.

Notice however that the above arguments rely on the hypothesis that the noise process is white and would no longer apply, should $\tilde{n}(t)$ possess an arbitrary correlation function. On the other hand, the process (3) is easily seen, based upon the scale property of linear systems, to be closed under linear transformations, in the sense that any linear operation leaves unaffected the marginal pdf of $\tilde{n}(t)$. Consequently, one might conceive to whiten the noise by linearly filtering the received signal: therefore, the problem of detecting the waveforms $\tilde{p}_i(t)$ in the presence of correlated disturbance can be reduced to that of detecting their filtered version, $\tilde{q}_i(t)$ say, in uncorrelated noise. However, upon filtering, these waveforms are no longer ensured to be equi-energy, whence such a test would no longer admit the detected envelopes as sufficient statistics: if, however, based upon the knowledge of the noise correlation, the transmitted waveforms are suitably scaled so as to ensure that $\tilde{q}_i(t) \quad i = 1, \ldots, M$ is a set of equi-energy signals, the test (10) is still *canonically* optimum.

ASYMPTOTIC PERFORMANCE

The analytical evaluation of the error probability for arbitrary signal constellation, subject to uncorrelated spherically symmetric impulsive noise, can take great advantage of some known results about detection in Gaussian noise. In fact, conditioned upon s, the noise process is a mere Gaussian one: thus, if we denote by $P(e|s)$ the error probability under such a disturbance, the unconditional performance is then obtained by simply averaging s out of $P(e|s)$.

The performance of the test (10) -for finite M's- subject to spherically symmetric uncorrelated noise has been assessed in [6,7,8] in the general case. Here we focus on the assessment of the asymptotic error probability: to this end, we first assume that the signal constellation consists of a set of M equally likely, orthogonal, equi-energy signals and then we let M diverge to infinity.

The conditional performance for finite M over non-fluctuating channel is expressed as [1]

$$P(e|s,A) = \sum_{k=1}^{M-1} \binom{M-1}{k} \frac{(-1)^{k+1}}{k+1} \exp\left(-\frac{A^2}{A_{rms}^2 s^2}\gamma_R \frac{k}{k+1}\right) \qquad (11)$$

wherein A_{rms} is the channel root mean square (rms) gain and γ_R is the signal-to-noise ratio per symbol, $\gamma_R = \mathcal{E}_p/\sigma^2$, related to the signal-to-noise ratio per bit γ_b as $\gamma_R = \gamma_b \log_2 M$. The error probability subject to impulsive, possibly fading channel is obviously obtained by averaging (11) with respect to A and s.

Starting upon relationship (11), the limit performance can be obtained by letting M diverge to infinity:

$$P_\infty(e|s,A) = \begin{cases} 1 & \text{if } \frac{\gamma_b A^2}{A_{rms}^2 s^2} < \ln 2 \\ 0 & \text{otherwise} \end{cases} \qquad (12)$$

which confirms that arbitrarily reliable communication on non-fluctuating Gaussian channel (i.e., for $A = A_{rms}$ and $s = 1$) can be achieved for any signal-to-noise ratio above the Shannon limit ln 2 (-1.6 dB), provided that the signal space has infinite dimensions.

In order to investigate the impact of impulsive noise and possible fluctuations in the signal amplitude, we are to average s and A out of (12). To begin with, assume that the channel is impulsive and deterministic. Averaging (12) with respect to s yields

$$\begin{aligned} P_\infty(e) &= \int_0^\infty u\left(s - \sqrt{\tfrac{\gamma_b}{\ln 2}}\right) f(s)\, ds \\ &= 1 - F_S\left(\sqrt{\tfrac{\gamma_b}{\ln 2}}\right) \end{aligned} \qquad (13)$$

wherein $u(\cdot)$ denotes the unit step function and $F_S(\cdot)$ is the cumulative distribution function of the random constant s. Relationship (13) highlights that impulsive noise results into smoothing of the Shannon limit curve, which no longer exhibits the stepwise transition between completely unreliable and fully reliable communication; conversely, one might expect that the heavier the tails

of the complementary cumulative distribution $1 - F_S(\cdot)$ - namely the spikier the noise -, the smoother the transition.

As working examples, we considered two cases of relevant practical interest, namely

- Compound-Gaussian noise with Generalised Laplacian marginal distribution, implying

$$f(s) = \frac{2\nu^{\nu}}{\Gamma(\nu)} s^{2\nu-1} \exp(-\nu s^2) \qquad (14)$$

- Compound-Gaussian noise with Middleton Class-A marginal distribution, implying

$$f(s) = \sum_i \epsilon_i \delta(s - s_i) \qquad (15)$$

where $\delta(\cdot)$ denotes the Dirac's delta function.

Correspondingly, the limit performance is written as

$$P_\infty(e) = 1 - \frac{1}{\Gamma(\nu)} \gamma\left(\nu; \frac{\nu\gamma_b}{\ln 2}\right) \qquad (16)$$

for the Generalised Laplace, and

$$P_\infty(e) = 1 - \sum_i \epsilon_i u\left(\sqrt{\frac{\gamma_b}{\ln 2}} - s_i\right) \qquad (17)$$

for the Middleton Class-A: in the above equations, $\gamma(\cdot;\cdot)$ is the incomplete Eulerian Gamma function.

Sample curves are reported in figures 2a and 2c, for Laplacian and Middleton Class-A marginal pdf, respectively (dot-dash curves): in the latter model, we assumed $\lambda = 0$, namely we nullified the Gaussian component in favour of the impulsive one. In both cases, decreasing values of the shape parameters - implying heavier and heavier tails in the noise distribution - result into smoother and smoother curves; conversely, as the shape parameters tend to infinity, namely as both the Generalised Laplacian and the Middleton Class-A distributions approach their Gaussian limit, the performance curves tend toward the well-known Shannon limit pertaining to Gaussian channels.

The curve referring to Middleton class-A pdf deserves some further comments. First, notice the stepwise behaviour of $P_\infty(e)$ with increasing signal-to-noise ratio per bit: this is obviously due to the discrete nature of the modulating random constant s. Somewhat surprising might appear the behaviour of $P_\infty(e)$ as γ_b vanishes, since the error probability does not tend to unity. In fact, it can be easily shown that, for Middleton class-A pdf, we have

$$\lim_{\gamma_b \to 0} P_\infty(e) = 1 - \epsilon_0 = 1 - e^{-\nu} \qquad (18)$$

This is a straightforward consequence of the neglect for the Gaussian component of the noise, namely of the assumption $\lambda = 0$: in fact, this implies a nonzero probability of no noise in the observed signal (see equations (8) and (15)).

In order to investigate the effect of the fading, we relax the hypothesis of non-fluctuating channel. Thus, a general expression for the limit performance subject to impulsive noise *and* fading can be achieved by averaging (12) with respect to A and s, which yields:

$$P_\infty(e) = 1 - \int_0^\infty F_S\left(\sqrt{\frac{\gamma_b}{\ln 2}} \frac{A}{A_{rms}}\right) f(A)\, dA \qquad (19)$$

The above relationship holds true for arbitrary fading and noise marginal distribution: on one hand, it represents a generalisation of (13), to which it reduces for the special case $f(A) = \delta(A - A_{rms})$; on the other, it also generalises a similar result recently established by Crepeau [9] with reference to white Gaussian noise. In fact, by inverting the order of integration, equation (19) can be re-cast in the form:

$$P_\infty(e) = 1 - \int_0^\infty F_A\left(sA_{rms}\sqrt{\frac{\ln 2}{\gamma_b}}\right) f(s)\, ds \qquad (20)$$

where now $F_A(\cdot)$ denotes the cumulative distribution function of the fading.

As a general comment, the main effect of the fading is seen to be a further smoothing of the limit curve: in other words, the deeper the fading, the slower the convergence to zero of $P_\infty(e)$.

In order to give a quantitative insight into the effect of the fading depth, we first have to select a model for the fluctuation law $f(A)$: a reasonable and quite general one is the so-called Nakagami-m family of distributions, i.e.:

$$f(A) = \frac{2m^m A^{2m-1}}{\Gamma(m) A_{rms}^{2m}} \exp\left[-m\left(\frac{A}{A_{rms}}\right)^2\right] \qquad (21)$$

where m is a shape parameter ruling the fading depth, and is precisely the ratio of the fading standard deviation to its root mean square value. In other words, the larger m, the more constrained the fluctuation: such a family subsumes some cases of relevant practical interest, such as the Rayleigh channel ($m = 1$), the one-plus-dominant Rayleigh ($m = 2$) and the non-fluctuating ($m = \infty$).

As worked example we again consider the two cases of spherically symmetric noise with Generalised Laplace and Middleton Class-A marginal pdf; for both cases, an explicit expression for the asymptotic performance in Nakagami-m fading channel can be derived by substituting the corresponding expressions for $F_S(\cdot)$ into (21),

which yields

$$P_\infty(e) = 1 - \frac{2m^m}{\Gamma(m)\Gamma(\nu)} \int_0^\infty \gamma\left(\nu; \frac{\nu\gamma_b}{\ln 2}x^2\right) x^{2m-1}e^{-mx^2}\,dx \qquad (22)$$

for the Generalised Laplace, and

$$P_\infty(e) = \sum_i \frac{\epsilon_i}{\Gamma(m)}\gamma\left(m; \frac{s_i^2 m \ln 2}{\gamma_b}\right) \qquad (23)$$

for the Middleton Class-A. In figures 2a and 2c the effect of the fading depth is represented for Laplacian and Middleton Class-A marginal pdf, respectively and for several values of the shape parameter ν: the limiting cases of non-fluctuating channel ($m = \infty$) and Rayleigh channel ($m = 1$) are reported only. As general comments, it is easily seen that, in both cases, the deeper the fading, the smoother the transition between unreliable and fully reliable communication. Further, for extremely deep fading, as, for example, observed on Rayleigh channels ($m = 1$), the asymptotic performance is primarily ruled by the fading law, and is practically unaffected by the particular noise model being in force. In order to further investigate the joint effect of the fading and of the noise spikyness, we refer to figures 2b and 2d, where, for the same noise distributions, the limiting performance is reported for several values of m and the two limiting situations of highly spiky (ν=0.1) and Gaussian ($\nu = \infty$) noise: as expected, for both noise models and for any implemented value of the shape parameter ν, the deeper the fading, the smoother the limit curve, which admits the two cases $m = \infty$ and $m = 1$ as lower and upper bound, respectively.

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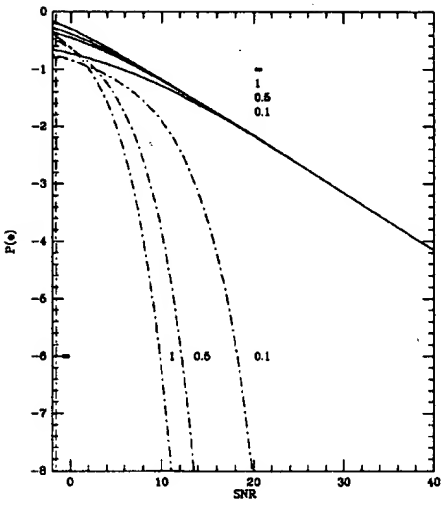

Figure 2a

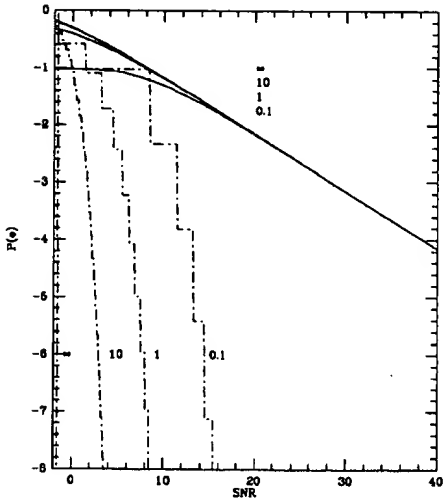

Figure 2c

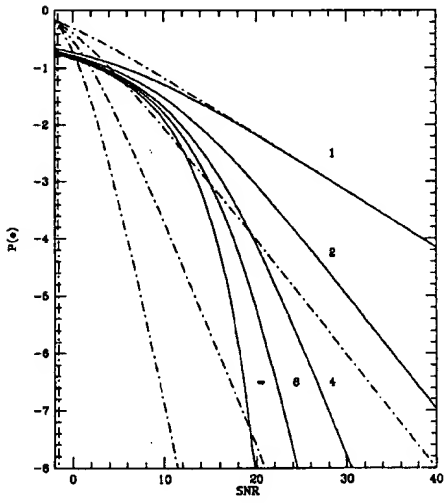

Figure 2b

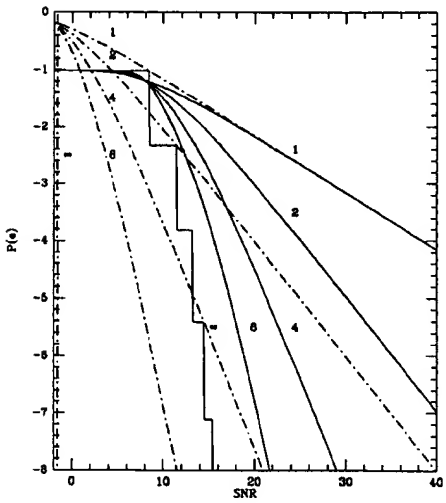

Figure 2d

Fig. 2 Asymptotic performance in impulsive noise.

a) Laplacian pdf for several ν's: —— Rayleigh fading, – · – · – non-fluctuating.
b) Laplacian pdf for several m's: —— $\nu = 0.1$, – · – · – $\nu = \infty$.
c) Middleton class-A pdf for several ν's: —— Rayleigh fading, – · – · – non fluctuating.
b) Middleton class-A pdf for several m's: —— $\nu = 0.1$, – · – · – $\nu = \infty$.